

A note on commutativity of rings with additive mappings

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Received 8 August 2012; revised 18 October 2012; accepted 19 October 2012

Available online 14 November 2012

Abstract. We investigate commutativity of the ring R involving some additive mapping with necessary torsion restrictions on commutators. We give counter examples which show that the hypotheses of our theorems are not superfluous.

Mathematical subject classification: 47B47; 16U80

Keywords: Additive mapping; Commutativity; Commutator

1. INTRODUCTION

This research is inspired by the work of Ashraf and Quadri [1,2]. Throughout this paper R will denote an associative ring with the identity 1. A ring R is said to be n -torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. An additive mapping $d: R \rightarrow R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. We say that a map $f: R \rightarrow R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for $x, y \in R$. In [3], Bell and Daif investigated a certain kind of commutativity preserving maps as follows: Let S be a subset of R . A map $f: S \rightarrow R$ is called strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. Precisely, they proved that if a semiprime ring R admits a derivation which is SCP on a right ideal ρ , then $\rho \subseteq Z(R)$. In particular, R is

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[☆] This research is partially supported by the research grants (UGC Grant No. 39-37/2010(SR)) and (INT/SLOVENIA/P-18/2009).

Peer review under responsibility of King Saud University.



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commutative if $\rho = R$. In [4], Deng and Ashraf proved that if there exist a derivation d of a semiprime ring R and a map $f: I \rightarrow R$ defined on a nonzero ideal I of R such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$.

The literature includes several papers on commutativity in rings with commutator constraints involving elements of the ring and images of elements under suitable maps (see [1, 5–8]). In this paper, our intent is to investigate the commutativity of rings satisfying certain identities involving additive mapping on the ring R . Throughout this paper we will denote $\max\{m, n\}$ by $(m \vee n)$, where m and n are positive integers.

2. THE MAIN RESULTS

We begin with the following lemma which is essential for developing the proof of our main results.

Lemma 2.1. *If there is a positive integer n such that $[x, y^n] = 0$ for all $x, y \in R$ and commutators in R are $n!$ -torsion free, then R is commutative.*

Proof. By the hypothesis, we have

$$[x, y^n] = 0 \text{ for all } x, y \in R. \quad (2.1)$$

Replacing y by $1 + y$ in (2.1), we obtain

$$\binom{n}{1}[x, y] + \binom{n}{2}[x, y^2] + \dots + \binom{n}{n-1}[x, y^{n-1}] = 0 \text{ for all } x, y \in R. \quad (2.2)$$

Substituting qy for y in (2.2), where $q = 1, 2, \dots, (n-1)$, we get

$$q \binom{n}{1}[x, y] + q^2 \binom{n}{2}[x, y^2] + \dots + q^{n-1} \binom{n}{n-1}[x, y^{n-1}] = 0 \text{ for all } x, y \in R. \quad (2.3)$$

The above equation produces the system of $(n-1)$ homogeneous equations, the coefficient matrix of this system is Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (n-1) & (n-1)^2 & \dots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n , it follows that $\binom{n}{r}[x, y^r] = 0$ for all $x, y \in R$ and $r = 1, 2, \dots, (n-1)$. In particular, for $r = 1$, we have $n[x, y] = 0$ for all $x, y \in R$. Since commutators in R are n -torsion free, the last expression implies that $[x, y] = 0$ for all $x, y \in R$. Hence, R is commutative. This proves the lemma. \square

Theorem 2.2. *Let $f: R \rightarrow R$ be any mapping. Then the following are equivalent:*

- (i) *There is an integer $n > 1$ such that $(f[x, y])^n = \pm [x^n, y^n]$ for all $x, y \in R$ and commutators in R are $n!$ -torsion free.*
- (ii) *There is an integer $n > 1$ such that $f([x, y]^n) = \pm [x^n, y^n]$ for all $x, y \in R$ and commutators in R are $n!$ -torsion free.*
- (iii) *There are positive integers m and n with $m + n > 2$ such that $(f^m[x, y])^n = \pm [x^m, y^n]$ for all $x, y \in R$ or $(f^n[x, y])^m = \pm [x^n, y^m]$ for all $x, y \in R$ and commutators in R are $(m \vee n)!$ -torsion free.*
- (iv) *There are positive integers m and n with $m + n > 2$ such that $f^m([x, y]^n) = \pm [x^m, y^n]$ for all $x, y \in R$ or $f^n([x, y]^m) = \pm [x^n, y^m]$ for all $x, y \in R$ and commutators in R are $(m \vee n)!$ -torsion free.*
- (v) *R is commutative.*

Proof. It is immediate that commutativity of R implies each of the conditions (i) through (iv). Now, we show that each of the conditions implies commutativity of R . (i) \Rightarrow (v).

We assume that

$$(f[x, y])^n = \pm [x^n, y^n] \text{ for all } x, y \in R. \quad (2.4)$$

Substituting x by $1 + x$ in (2.4), we get

$$(f[x, y])^n = \pm \binom{n}{1} [x, y^n] \pm \binom{n}{2} [x^2, y^n] \pm \dots \pm \binom{n}{n-1} [x^{n-1}, y^n] \pm [x^n, y^n] \text{ for all } x, y \in R.$$

Application of (2.4) yields that

$$\binom{n}{1} [x, y^n] + \binom{n}{2} [x^2, y^n] + \dots + \binom{n}{n-1} [x^{n-1}, y^n] = 0 \text{ for all } x, y \in R. \quad (2.5)$$

Now, using the same techniques as we have used in Lemma 2.1, we get $[x, y^n] = 0$ for all $x, y \in R$. Thus, in view of Lemma 2.1, we conclude that R is commutative.

(ii) \Rightarrow (v) is similar to (i) \Rightarrow (v).

(iii) \Rightarrow (v). First we consider the case

$$(f^m[x, y])^n = \pm [x^m, y^n] \text{ for all } x, y \in R. \quad (2.6)$$

Replacing x by $1 + x$ in (2.6), we get

$$\binom{m}{1} [x, y^n] + \binom{m}{2} [x^2, y^n] + \dots + \binom{m}{m-1} [x^{m-1}, y^n] = 0 \text{ for all } x, y \in R. \quad (2.7)$$

Using the same arguments as we have used to prove (i) \Rightarrow (v), we conclude that

$\binom{m}{r} [x^r, y^n] = 0$ for all $x, y \in R$ and $r = 1, 2, \dots, (m-1)$. In particular, for $r = 1$ we have $m[x, y^n] = 0$ for all $x, y \in R$. The last expression implies that $[x, y^n] = 0$ for

all $x, y \in R$, since commutators in R are m -torsion free. Thus, by Lemma 2.1, R is commutative.

Similar conclusion holds for the case $(f^m[x, y])^n = \pm [x^n, y^m]$ for all $x, y \in R$.

(iv) \Rightarrow (v). Use the parallel arguments as we have used in the proof of (iii) \Rightarrow (v). This completes the proof of the theorem. \square

Theorem 2.3. *Let $f, g: R \rightarrow R$ be two mappings such that f is additive and $f(1) = 1$. Then the following are equivalent:*

- (i) *There is an integer $n > 1$ such that $[f(x), g(y)]^n = \pm [x^n, y^n]$ for all $x, y \in R$, and commutators in R are $n!$ -torsion free.*
- (ii) *There are positive integers m and n with $m + n > 2$ such that $[f^m(x), g^n(y)] = \pm [x^m, y^n]$ for all $x, y \in R$ or $[f^m(x), g^n(y)] = \pm [x^n, y^m]$ for all $x, y \in R$, and commutators in R are $(m \vee n)!$ -torsion free.*
- (iii) *There are positive integers m and n with $m + n > 2$ such that $[f^m(x), g^m(y)]^n = \pm [x^m, y^n]$ for all $x, y \in R$ or $[f^m(x), g^m(y)]^n = \pm [x^n, y^m]$ for all $x, y \in R$, and commutators in R are $(m \vee n)!$ -torsion free.*
- (iv) *R is commutative.*

Proof. Clearly, (iv) \Rightarrow (i), (iv) \Rightarrow (ii) and (iv) \Rightarrow (iii). Now, we will prove that

(i) \Rightarrow (iv). By the assumption, we have

$$[f(x), g(y)]^n = \pm [x^n, y^n] \text{ for all } x, y \in R. \quad (2.8)$$

Replacing x by $1 + x$ in (2.8), we get

$$[f(1) + f(x), g(y)]^n = \pm \binom{n}{1} [x, y^n] \pm \binom{n}{2} [x^2, y^n] \pm \dots \pm \binom{n}{n-1} [x^{n-1}, y^n] \pm [x^n, y^n]$$

for all $x, y \in R$. Using (2.8) and the fact that image of identity is identity under f , we conclude that

$$\binom{n}{1} [x, y^n] + \binom{n}{2} [x^2, y^n] + \dots + \binom{n}{n-1} [x^{n-1}, y^n] = 0 \text{ for all } x, y \in R.$$

Using parallel arguments as we have used to prove (i) \Rightarrow (v) in Theorem 2.2, we find that $\binom{n}{r} [x^r, y^n] = 0$ for all $x, y \in R$ and $r = 1, 2, \dots, (n-1)$. In particular, we have $n[x, y^n] = 0$ for all $x, y \in R$. Since commutators in R are n -torsion free, the last expression yields that $[x, y^n] = 0$ for all $x, y \in R$. Hence, R is commutative by Lemma 2.1.(ii) \Rightarrow (iv).

First we assume that

$$[f^m(x), g^n(y)] = \pm [x^m, y^n] \text{ for all } x, y \in R. \quad (2.9)$$

Replacing x by $1 + x$ in (2.9), we obtain

$$[f^m(1+x), g^n(y)] = \pm \binom{m}{1} [x, y^n] \pm \binom{m}{2} [x^2, y^n] \pm \dots \pm \binom{m}{m-1} [x^{m-1}, y^n] \pm [x^m, y^n]$$

for all $x, y \in R$. Using (2.9) and the fact that the image of identity is identity under f , we get

$$\binom{m}{1}[x, y^n] + \binom{m}{2}[x^2, y^n] + \dots + \binom{m}{m-1}[x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.$$

The above expression is similar to the relation (2.7) and henceforth using the same approach as we have used to obtain commutativity of R from the expression (2.7) in the proof of Theorem 2.2, we get the required result.

Similarly, we can prove the result for the case $[f^m(x), g^n(y)] = \pm [x^n, y^m]$ for all $x, y \in R$.

(iii) \Rightarrow (iv). It can be proved by using the same techniques with necessary variations. Thereby, the proof is completed. \square

The next theorem is motivated by [4, Theorem 1].

Theorem 2.4. *Let $d: R \rightarrow R$ be a derivation of R and g be any mapping of R . If there are positive integers m and n with $m + n > 2$ such that $[d(x)^m, g(y)^n] = [x^m, y^n]$ for all $x, y \in R$ and commutators in R are $(m \vee n)!$ -torsion free, then R is commutative.*

Proof. By the assumption, we have

$$[d(x)^m, g(y)^n] = [x^m, y^n] \text{ for all } x, y \in R. \quad (2.10)$$

Replacing x by $1 + x$ in (2.10) and using the fact that $d(1) = 0$, we get

$$\binom{m}{1}[x, y^n] + \binom{m}{2}[x^2, y^n] + \dots + \binom{m}{m-1}[x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.$$

The above expression is same as the relation in (2.7) and henceforth using the same approach as we have used to obtain commutativity of R from the expression (2.7) in the proof of Theorem 2.2, we get the required result. This proves the theorem. \square

At the end, let us write two examples which show that the restriction in our results are not superfluous.

Example 2.1. Let S be any noncommutative ring and

$$R = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} \middle| a, b \in S \right\}.$$

Obviously, R is a ring without identity. Also, it can be easily seen that for any integer $n > 1$, the identity $[x, y^n] = 0$ holds for all $x, y \in R$, but R is not commutative. Hence, in Lemma 2.1 identity element is necessary. Further, define a mapping $f: R \rightarrow R$ such that

$$f \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \quad a, b \in S.$$

It is easy to see that f satisfies all the requirements of Theorem 2.2. However, R is not commutative.

Example 2.2. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Clearly, R is a ring without identity. Consider the mappings $f, g: R \rightarrow R$ such that

$$f \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{Z}.$$

It is straightforward to check that f and g satisfy all the requirements of Theorem 2.3, but R is not commutative.

ACKNOWLEDGMENTS

The authors are greatly indebted to the learned referee for his/her valuable suggestions which improved the paper immensely.

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